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Is the Zamolodchikov model critical?

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Abstract. Evidence is presented in favour of the hypothesis that the Zamolodchikov model (an exactly solvable three-dimensional lattice model without any temperature-like parameters) is critical. The evidence is obtained by generalising the Zamolodchikov model to include a temperature-like variable. The magnetisation curve of this model is then studied using a modified form of a variational approximation formulated earlier. Also we show the two-layer Zamolodchikov model corresponds to a critical free-fermion model.

1. Introduction

The Zamolodchikov model (Zamolodchikov 1981) is unique amongst three-dimensional lattice models in statistical mechanics for two reasons. First, as conjectured by Zamolodchikov (1981) and subsequently proved by Baxter (1983), the model permits a solution of the tetrahedron equations. These are a set of 2^{14} equations sufficient for the plane-to-plane transfer matrices to commute. Secondly, as recently shown by Baxter (1984), the free energy of the model can be calculated exactly.

The Zamolodchikov model does not have any temperature-like parameters. In two-dimensional lattice statistics there is a class of models which are exactly solvable but only at a special temperature—the q-state Potts model at criticality (Baxter 1982). This immediately raises the question of whether the Zamolodchikov model is critical. In this paper we investigate that question.

To do this we define a generalised Zamolodchikov model. This model has a temperature-like parameter u, the original Zamolodchikov model corresponding to a special value $u = u_c$. By adapting a variational approximation for three-dimensional lattice models formulated earlier (Baxter and Forrester 1984) we obtain numerical results for this generalised model. In particular we calculate the magnetisation. Although our results are not conclusive they are consistent with the hypothesis that the model is critical at $u = u_c$.

Further evidence in support of this hypothesis is obtained by studying the two-layer Zamolodchikov model. We show it corresponds to a critical free-fermion model.

2. A variational approximation for the generalised Zamolodchikov model

2.1. The generalised Zamolodchikov model and an equivalent formulation

Let $\mathscr{L}$ be the simple cubic lattice. With each site i associate a spin σ_i, with values $+1$ or -1 (+ or −). Allow interactions between the eight spins round each elementary

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cube. The Zamolodchikov model has been formulated as such an 'interaction-round-a-cube' model by Baxter (1983), table 1. The weight function is invariant under both reflection and rotation (i.e. it is isotropic) when the spherical angles α_i equal $\frac{1}{4}\pi$, $i = 1, 2, 3, 4$. Explicitly, in the notation of Baxter (1983) the model is isotropic when the parameters assume the values

$$\begin{aligned} &P_0 = 1, && Q_0 = u^2 \\ &P_i = Q_i = u && i = 1, 2, 3 \\ &R_i = (u + u^3)^{1/2} && i = 0, 1, 2, 3 \end{aligned} \tag{2.1}$$

where

$$u = \tan(\pi/8) = \sqrt{2} - 1. \tag{2.2}$$

We can construct a sub-lattice $\mathscr{L}_s$ of $\mathscr{L}$ consisting of a particular site and all other sites that can be visited from it by walking along body diagonals of $\mathscr{L}$. Then $\mathscr{L}_s$ contains one-quarter of the sites of $\mathscr{L}$ and is in fact a BCC lattice where the unit cell consists of eight cubes of $\mathscr{L}$ (see figure 1).

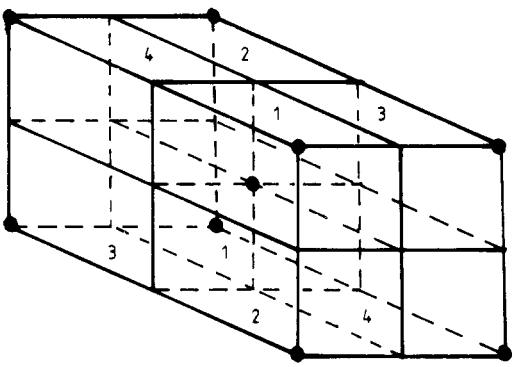

Figure 1. The four types of cubes 1, 2, 3, 4 obtained by negating the spins on the sites indicated by dots. The corresponding weight functions are W_1, W_2, W_3, W_4 respectively. The lattice formed by the dotted sites is a BCC.

We generalise the Zamolodchikov model by considering u in (2.1) as a variable:

$$0 < u < 1. \tag{2.3}$$

Then u is a temperature-like variable. For $u \simeq 0$ a dominant ground state is one in which the spins on $\mathscr{L}_s$ are down, the rest are up.

In Baxter and Forrester (1984) we formulated a variational approximation applicable to isotropic three-dimensional cubic lattice models which have a translation invariant ground state. Thus we cannot directly apply the variational principle to the generalised Zamolodchikov model. However the ground state can be made translation invariant by the following transformation.

Suppose we negate the spins on $\mathscr{L}_s$. We then have a partition function of the four types of cubes and weight functions indicated in figure 1. Using the notation for the location of spins on a cube given in figure 2 these weight functions are specified by table 1.

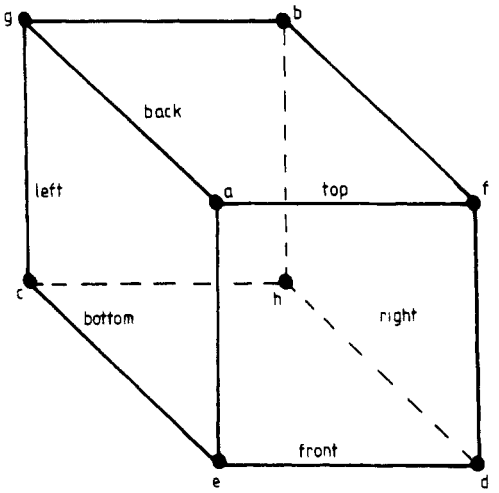

Figure 2. Labelling of the faces of a cube and positions of the spins in the weight $W(a|efg|bcd|h)$.

Table 1. Weight functions for the four types of cubes indicated in figure 1. Here we have used the notation $\lambda = abeh$, $\mu = acfh$, $\nu = adgh$, $L = u(1-abcd)$

λ	μ	ν	W_1	W_2	W_3	W_4
+	+	+	$1+abcdu^2$	$1+abcdu^2$	$1+abcdu^2$	$1+abcdu^2$
−	+	+	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$
+	−	+	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$
+	+	−	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$
+	−	−	$-abL$	$-abL$	abL	abL
−	+	−	$-acL$	acL	$-acL$	acL
−	−	+	$-adL$	adL	adL	$-adL$
−	−	−	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$	$(u+u^3)^{1/2}$

We define the sign factor

$$(a;b) = \begin{cases} -1 & \text{if } a = b = -1 \\ 1 & \text{otherwise} \end{cases} \tag{2.4}$$

which has the factorisation property

$$(ab;cd) = (a;c)(a;d)(b;c)(b;d). \tag{2.5}$$

Then from table 1 (with λ, μ, ν defined therein) we can check the relations

$$W_2 = (\lambda;\mu\nu)W_1, \qquad W_3 = (\mu,\lambda\nu)W_1, \qquad W_4 = (\nu,\lambda\mu)W_1. \tag{2.6}$$

Using the factorisation property (2.5) we can check that for each cube $j = 2, 3, 4$

$$W_j = W_1 \times \begin{pmatrix} \text{product of sign factors for each face of the} \\ \text{cube not bordering a type-1 cube} \end{pmatrix}. \tag{2.7}$$

The sign factor of a face with spins σ_1, σ_2, σ_3, σ_4 ordered anti-clockwise as in figure 3 and with the type-1 cube bordering the edges with spins (σ_1, σ_2) and (σ_3, σ_4) is given by

$$(\sigma_1\sigma_3; \sigma_2\sigma_4). \tag{2.8}$$

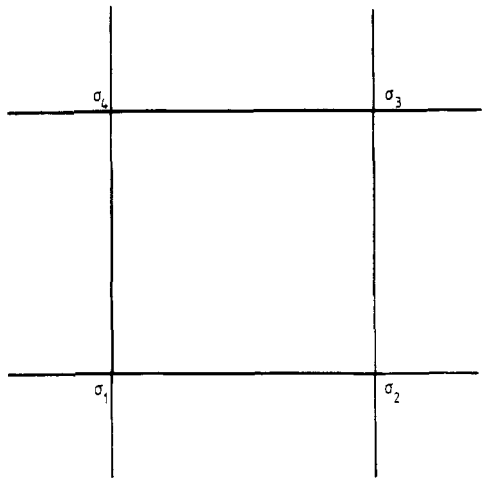

Figure 3. Ordering of the spins σ_1, σ_2, σ_3, σ_4 around a face. This same ordering is used for the arguments of the functions $F(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$.

Thus all sign factors are on faces of cubes not bordering a type-1 cube. Multiplying out the sign factors on any one such face, we find the total sign factor is

$$(\sigma_1; \sigma_2)(\sigma_2; \sigma_3)(\sigma_3; \sigma_4)(\sigma_4; \sigma_1) \tag{2.9}$$

which is a product of sign factors of each edge of the face. However, every edge of the lattice borders one cube of each type. Thus for each edge (with spins a, b say) there are two faces containing that edge not bordering a type-1 cube. Hence the sign factors for each edge occur in pairs and do not contribute to the partition function since

$$(a; b)^2 = 1. \tag{2.10}$$

The partition function is therefore unchanged if we assign the weight function W_1 to *all* cubes of $\mathcal{L}$. The model with this weight function has as ground state the state with all spins up, which is of course translation invariant.

2.2. A variational approximation for anisotropic models with a translation invariant ground state

The price we pay for using the weight function W_1 is to no longer have an isotropic weight function. Thus, for example, W_1 is not unchanged by plane reversal but rather for some configurations there is a change in sign (the Zamolodchikov model is not physical in the sense that there are negative Boltzmann weights). Therefore again we cannot directly use the variational approximation. However, by interpreting the variational expression for κ graphically, we can write down a variational approximation for anisotropic models.

We recall from Baxter and Forrester (1984) the variational quantities A, B and F can be interpreted as corner, edge and face weights respectively. In a variational approximation for anisotropic models A, B and F will depend on the particular corner, edge and face of the central cube. Let us label the faces of a cube front (f), back (b), left (l), right (r), top (t) and bottom (g, for 'ground'). Then the particular A's, B's and F's can be distinguished using these labels as subscripts. We require three such labels for A, two for B and one for F.

This distinguishes directions and orientations in the quantities s_i which constitute the variational approximation for κ (equation 3.18 of Baxter and Forrester 1984). There are now three different expressions for s_2 (s_{21}, s_{22}, s_{23} say) and three different expressions for $s_3(s_{31}, s_{32}, s_{33})$. When represented graphically s_{21}, s_{22} and s_{23} only differ by orientations and similarly s_{31}, s_{32}, s_{33} (thus in the isotropic case $s_{21} = s_{22} = s_{23}$, $s_{31} = s_{32} = s_{33}$). The variational expression for κ in the anisotropic case is

$$\kappa = s_4 s_{21} s_{22} s_{23} / (s_{31} s_{32} s_{33} s_1). \tag{2.11}$$

In figure 4 we represent s_4, s_{31}, s_{21} and s_1 graphically. We adopt the convention that the spins on dotted sites are to be summed over while sites with circles have the spins thereon as variables. The graphs are highly abbreviated. In the graph for s_4, with every corner, edge and face there is associated an A, B and F respectively, with the appropriate position subscripts. The graph for s_{31} contains two corner, edge and face variational weights, for each corner, edge and face of the graph; one will always have a subscript b for bottom, the other t for top. The graph for s_{21} contains four corner and edge weights for each corner and edge (the corners in this graph being defined as the ends of the edge), and the graph for s_1 contains the eight corner weights A. The graphs for s_{22} and s_{23} are obtained from that for s_{21} by orientating the graph (which is along the Y axis) along the X and Z axis respectively. The graphs for s_{32} and s_{33} are obtained from that for s_{31} by orientating the graph (which is in the XY plane) in the XZ and YZ planes respectively.

Similarly the three sets of variational equations determining the A's, B's and F's given in Baxter and Forrester (1984) (3.23)-(3.25) can be represented graphically. A typical graph of each of the three sets of variational equations is given in figure 5 (in

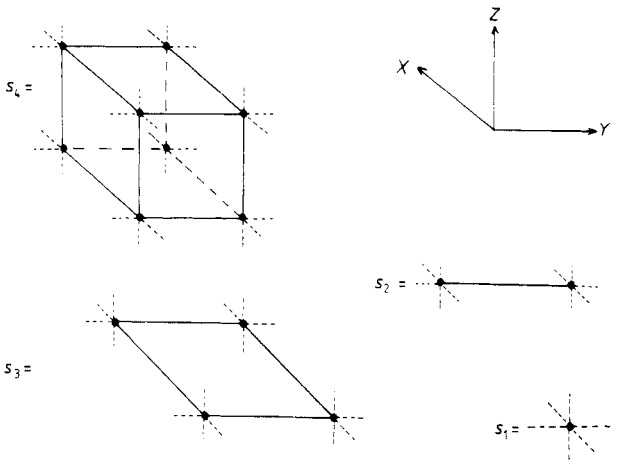

Figure 4. Graphical representation of s_4, s_{31}, s_{21} and s_1. See the text for explanations of these graphs.

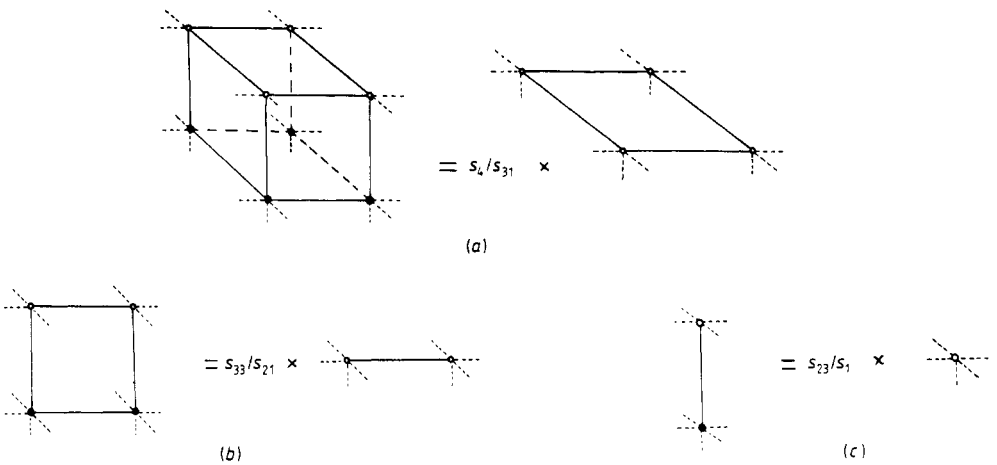

Figure 5. Graphical representation of the three types of variational equations. See the text for further explanation.

fact the type-(a) graph of figure 5 corresponds to (3.23) after cancellation of common factors on both sides of that equation). Again we are using an abbreviated notation. The number of variational quantities associated with each graph is the same as in figure 4 except that all variational quantities involving the top subscript t are not present. Due to the anisotropy there are 6×16 distinct equations of type-(a), 12×4 of type-(b) and 6×2 of type-(c) (the multiplicity coming from the allowed values of the arguments of F, B and A respectively), which can be obtained from those given in figure 5 by rotations.

Note that in general we have $8\times2+12\times4+6\times16$ variables so it appears we have less equations than unknowns. However, it is readily seen from the equation in figure 5(b) and the three equations it generates by rotations of 90° in the XZ plane that the quantities B occur in the combinations

$$B_{\mathrm{ft}}B_{\mathrm{bt}}, \qquad B_{\mathrm{ft}}B_{\mathrm{fg}}, \qquad B_{\mathrm{fg}}B_{\mathrm{bg}}, \qquad B_{\mathrm{bg}}B_{\mathrm{bt}}. \tag{2.12}$$

A knowledge of any three of the above determines the fourth, so in this block of four B's one is redundant. The same considerations apply to the set of B's

$$B_{\mathrm{rt}}, \qquad B_{\mathrm{rg}}, \qquad B_{\mathrm{lg}}, \qquad B_{\mathrm{lt}} \tag{2.13}$$

and

$$B_{\mathrm{fl}}, \qquad B_{\mathrm{fr}}, \qquad B_{\mathrm{bl}}, \qquad B_{\mathrm{br}} \tag{2.14}$$

so out of the original twelve B's only nine are distinct. Similarly the A's only occur in the combinations

$$A_{\mathrm{brt}}A_{\mathrm{brg}}, \qquad A_{\mathrm{brt}}A_{\mathrm{blt}}, \qquad A_{\mathrm{blt}}A_{\mathrm{blg}}, \qquad A_{\mathrm{blg}}A_{\mathrm{brg}} \tag{2.15}$$

and

$$A_{\mathrm{frt}}A_{\mathrm{frg}}, \qquad A_{\mathrm{frt}}A_{\mathrm{flt}}, \qquad A_{\mathrm{flt}}A_{\mathrm{flg}}, \qquad A_{\mathrm{flg}}A_{\mathrm{frg}}. \tag{2.16}$$

Out of the four variables given by (2.15) only three are distinct and similarly the four variables given by (2.16). Thus only six of the original eight A's are distinct. Hence we have in fact more equations than unknowns.

2.3. General first-order solutions of the variational equations

Firstly let us define a normalised weight function $\bar{W}_1$ by

$$\bar{W}_1 = W_1 / W_1(+|+++|+++|+) \tag{2.17}$$

and suppose that the equation in figure 5(a) is written in terms of $\bar{W}_1$. When the system is in its ground state of all spins up the solution of the above specified variational equations is, for each $i, j = 1, 2, 3$

$$s_4/s_{3i} = s_{3i}/s_{2j} = s_{2j}/s_1 = 1, \tag{2.18}$$

for each of the eight A's

$$A(\sigma_i) = \delta(\sigma_i, +), \tag{2.19}$$

for each of the twelve B's

$$B(\sigma_i, \sigma_j) = \delta(\sigma_i, +)\delta(\sigma_j, +), \tag{2.20}$$

and for each of the six F's

$$F(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = \delta(\sigma_i, +)\delta(\sigma_j, +)\delta(\sigma_k, +)\delta(\sigma_l, +). \tag{2.21}$$

Here δ denotes the Kronecker delta.

We know from our study of the variational equations in the isotropic case that in the low-temperature limit the term with the spins in the summation on the LHS all equal to $+1$ dominates and is equal in magnitude to the RHS. To see that this is true in general divide each term on the LHS by the RHS, and cancel out common factors. If we then substitute the ground state values (2.18)-(2.21) all terms are zero except that with all spins in the summation of the LHS equal to $+1$, which is not well defined (we have $0/0$). As a first-order approximation to the variational equations we equate this term for each equation to unity.

We seek a solution to these first-order equations. Again we know from the isotropic case a solution can be obtained by considering the blocks of spins corresponding to the A's, B's and F's now as single cubes of spins, fixed to the appropriate position around a central cube. The spins on the outside (i.e. not bordering the central cube) are fixed at $+1$, and the spins bordering the central cube are variables. By assigning these single cubes or spins their weight function $\bar{W}_1$, we obtain a solution of the first-order variational equations. This is again a solution of the first-order equations in the anisotropic case, a fact which is readily seen by examining the graphical interpretation of these equations.

For example we have to first order

$$A_{\mathrm{frg}}(-) = \bar{W}_1(+|++-|+++|+), \qquad B_{\mathrm{rt}}(+,-) = \bar{W}_1(+|-++|+-+|+) \tag{2.22}$$
$$F_{\mathrm{t}}(-,+,-,+) = \bar{W}_1(+|-++|+++|-).$$

Note that in the ordering of the arguments of the F's we adopt the convention of figure 3.

2.4. Ansatz for the reduction of the number of independent variables

The results of §§ 2.2 and 2.3 are applicable to any anisotropic cubic lattice model with a translation invariant ground state. However, due to the large number of independent

variables the general formalism is of no practical use. We thus seek a procedure to reduce the number of independent variables to a manageable level.

If W_1 were isotropic we could use the obvious symmetries of the A's, B's and F's to reduce the number of distinct equations and unknowns to 11, regaining the equations of Baxter and Forrester (1984). In fact W_1 is 'almost isotropic' in the sense that spatial rotations and reflections leave unchanged the magnitudes of the values of W_1, but may change their sign. We may hope that these symmetries and anti-symmetries are reflected in our variational equations. We find that indeed they are, and that we can reduce the number of equations to eleven, but we do have to be careful to obtain the correct signs.

The ansatz we use is as follows. If the variational quantities (in particular the blocks of spins corresponding to the variational quantities) differ only by orientation, and if the corresponding first-order solutions differ by at most a sign, then these quantities are equated (with the appropriate sign). For example we have

$$F_t(+,+,-,-) = \bar{W}_1(+|{+}{+}{+}|{+}{-}{+}|-) = -2u/(1+u^2) \tag{2.23}$$

$$F_t(-,+,+,-) = \bar{W}_1(+|{-}{+}{+}|{+}{-}{+}|+) = 2u/(1+u^2) \tag{2.24}$$

so we substitute

$$F_t(+,+,-,-) = -F_t(-,+,+,-) \tag{2.25}$$

in the variational equations.

We thus find for any orientation

$$A = B = F = 1 \tag{2.26}$$

when all the arguments are + (this is a normalisation). Also each of

$$A(-), \quad B(+,-), \quad F(+,+,+,-), \quad F(-,-,-,+), \quad F(-,-,-,-) \tag{2.27}$$

is independent of orientation. The quantities

$$B(-,-) \qquad \text{and} \qquad F(+,+,-,-) \tag{2.28}$$

are dependent on orientation. The sign of these quantities is dependent on location of the $-$ spins on the relvant edge and face. If $B(-,-)$ is the variational quantity associated with an edge containing the spins a or h (recall figure 2) then we have

$$B(-,-) = B'. \tag{2.29}$$

Otherwise

$$B(-,-) = -B'. \tag{2.30}$$

Similarly if $F(+,+,-,-)$ (or any F obtained from this F by rotation) is the variational quantity associated with a face of the central cube containing the spins a or h and if a or h (respectively) are $-$ then

$$F(+,+,-,-) = F'. \tag{2.31}$$

Otherwise

$$F(+,+,-,-) = -F'. \tag{2.32}$$

However our first-order rule gives

$$F(+,-,+,-) = 0 \tag{2.33}$$

for each of the six F's and those F's obtained from them by rotations. Thus these quantities are of the same magnitude but their signs are undetermined. To determine the signs it is necessary to solve the appropriate variational equation (figure 5(a)) to next order. This is done by equating the RHS to the next dominant term(s) on the LHS in the low-temperature limit (which are the terms with all but one spin +1 in the summation). $F(+,-,+,-)$ is considered as the unknown, which is determined by substituting the first-order values of the other terms. We thus find to second order

$$F(+,-,+,-) = \varepsilon \times 2\bar{W}^3(+|++-|-++|+)\,\bar{W}^2(+|-++|+++|+)$$
$$= \varepsilon \times 16u^4/(1+u^2)^4, \tag{2.34}$$

where the sign factor ε is specified in the same way as the sign of $F(+,+,-,-)$. Thus if $F(+,-,+,-)$ is the variational quantity associated with a face of the central cube containing the spins a or h and if a or h (respectively) are -1 then $\varepsilon = 1$. Otherwise $\varepsilon = -1$. The same is true for any F obtained from $F(+,-,+,-)$ by rotation.

Substitution of these relationships between the variational variables shows

$$s_{31} = s_{32} = s_{33} \equiv s_3 \tag{2.35}$$

and

$$s_{21} = s_{22} = s_{23} \equiv s_2. \tag{2.36}$$

There are thus eleven unknowns (we take $A(+) = B(+) = F(+,+,+,+) = 1$ as normalisations and s_4/s_3, s_3/s_2, s_2/s_1 are variables) just as in the isotropic case.

It remains to check that there are exactly eleven independent variational equations (which are, apart from signs, the same as in the isotropic case). This is equivalent to requiring each term in the expression for s_4 to be isotropic. By this we mean each configuration that is related to another by a rotation or reflection, when weighted by the variational quantities as in the definition of s_4, must be equal. We readily verify this.

In fact the equations reduce to those of an isotropic translation invariant model, (3.18)-(3.25) of Baxter and Forrester (1984), with W therein replaced by $|\bar{W}_1|$, except for certain weights which occur with a negative sign. These exceptional weights are those corresponding to the spin configurations shown in figure 6 and to any configuration obtainable therefrom by spatial reflection and/or rotation. Because of these negative signs, we lose the original spin-negation symmetries of the Zamolodchikov model.

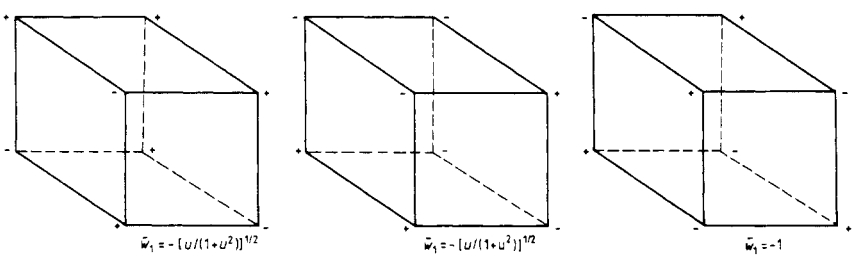

Figure 6. Spin configurations that have negative weight in the final eleven-equation variational approximation. There are 8 configurations that can be obtained from the first by spatial reflection and/or rotation; 24 from the second and two from the third: all these have negative weight.

2.5. Series and numerical results

The variational equations can be solved both numerically and in series form using the procedure outlined in Baxter and Forrester (1984), where we obtained such results for the Ising models. If κ is the partition function per site of the generalised Zamolodchikov model (with weight function W_1) we obtain

$$\kappa/(1+u^2) = 1 + u^4 - 4u^6 + 4u^7 + 45u^8 - 36u^9 - 80u^{10} + 444u^{11} - 435u^{12} - 1888u^{13} + 13460u^{14}. \tag{2.37}$$

Also, the spontaneous magnetisation is given by

$$M_0 = [A^8(+) - A^8(-)]/[A^8(+) + A^8(-)], \tag{2.38}$$

and we find that

$$M_0 = 1 - 2u^4 + 8u^6 - 16u^7 - 158u^8 + 144u^9 - 176u^{10} - 2416^{11} + 3054^{12} + 6800u^{13} - 103288u^{14}. \tag{2.39}$$

For the SC, FCC and BCC Ising models we found that the variational approximation gave κ and M_0 correctly to orders u^{14}, u^{19}, u^{23} respectively. Unfortunately we have no direct test of the accuracy of the approximation for the Zamolodchikov model, but it seems likely that the series (2.37) and (2.39) are correct to the order given. (Even if the last coefficients are in error, we should expect the error to be small.)

2.6. Behaviour of $A(-)/A(+)$

In (2.26) we have normalised so that $A(+) = 1$, so $A(-)$ is the same as the ratio $A(-)/A(+)$, which from (2.38) is related to the spontaneous magnetisation. We have plotted $A(-)$ against u in figure 7.

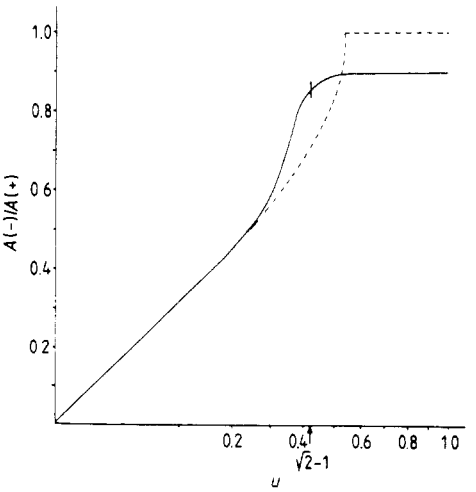

Figure 7. Graph of $A(-)$ against u for the BCC Ising model (broken lines) and the generalised Zamolodchikov model (full line). The scale on the u axis is proportional to $u^{1/2}$, although we have marked in the values of u. On the latter we have marked $u = \sqrt{2} - 1$.

First consider the behaviour of $A(-)$ for an Ising model: a typical graph is given in figure 7. As u increases $A(-)$ increases until a certain critical value u_c is reached (which occurs when $A(-)=A(+)=1$), and then $A(-)$ remains constant. In fact for the Ising models all the variational quantities increase monotonically until the critical value is reached, when they remain constant ('stick') for further increases in u. Further at criticality all variational quantities related by spin reversal are numerically equal, so for example

$$F(+,+,-,-)=F(-,-,+,+). \tag{2.40}$$

The plot of $A(-)$ against u for the generalised Zamolodchikov model is not identical in structure to that for the Ising models. In particular the value of $A(-)$ at which $A(-)$ remains constant is not $A(-)=1$. This is because we have lost the spin-negation symmetry in making the variational approximation. It can be explained by the anti-symmetry relations built into the variational approximation (recall § 2.4) where for certain faces of the cube we have for example

$$F(+,+,-,-)=-F(-,-,+,+). \tag{2.41}$$

At criticality we require the spin reversal symmetry (2.40) which thus implies

$$F(+,+,-,-)=0. \tag{2.42}$$

However, again the numerical values of the variational quantities are monotone, so it is not possible to obtain the usual solution of the variational equations at criticality, starting from the low-temperature solution and varying the solution continuously.

Even so, figure 7 does show the characteristic 'sticking' of the value of $A(-)$, which is indicative of a phase transition with vanishing spontaneous magnetisation (and the value of $A(-)$ is then quite close to one). We are particularly interested in the value $u=\sqrt{2}-1$, when from (2.2) we regain the original isotropic Zamolodchikov model. From figure 7 it is at about this value that the 'sticking' of $A(-)$ occurs. Thus our numerical results are consistent with the expectation that the generalised model is critical at

$$u_c=\sqrt{2}-1. \tag{2.43}$$

Guttmann (1984) has applied numerical series analysis techniques to (2.37) and (2.39). The results indicate a critical singularity between 0.3 and 0.4. Allowing for the shortness of the series, this is reasonably consistent with the conjecture (2.43).

2.7. Partition function per site

Very recently, one of us (Baxter 1984) has exactly evaluated the partition function per site of the Zamolodchikov model. For the isotropic case considered here the result is

$$\kappa=2^{3/4}(2^{1/2}-1)\,e^{2G/\pi}=1.2480\ldots \tag{2.44}$$

where $G=0.915\,965\ldots$ is Catalan's constant. This provides a good test of our numerical approximation, which gives (for $u=\sqrt{2}-1$)

$$\kappa=1.2333\ldots\,. \tag{2.45}$$

When one considers the simple nature of the variational approximation, and the fact that it is a severe test to apply it at a critical point, this is really very good agreement.

3. The two-layer case

Le us go back to the Zamolodchikov model as defined in table 1 of Baxter (1983), and for the moment regard the P_i, Q_i, R_i therein as adjustable parameters. This model is more general even than that discussed in § 2 (it reduces to that of § 2 when (2.1) is satisfied). Here we shall show that for a simple cubic lattice $\mathscr{L}$ of only two layers with cyclic boundary conditions, the general model is equivalent to a planar square-lattice eight-vertex model in a field. Further, when P_i, Q_i, R_i take the Zamolodchikov model values, the eight-vertex model reduces to a critical free-fermion model.

We consider two cubes of $\mathscr{L}$, one above the other and think of them as a 'pillar'. As $\mathscr{L}$ is only two layers thick and has cyclic boundary conditions, the spins at the base of the lower cube must be the same as those at the top of the upper, as in figure 8. Labelling the spins as in figure 8, the Boltzmann weight of the pillar is

$$W_p = W(a|a'bc|dc'b'|d')\,W(a'|ab'c'|d'cb|d), \tag{3.1}$$

where W is the single-cube weight function defined in table 1 of Baxter (1983).

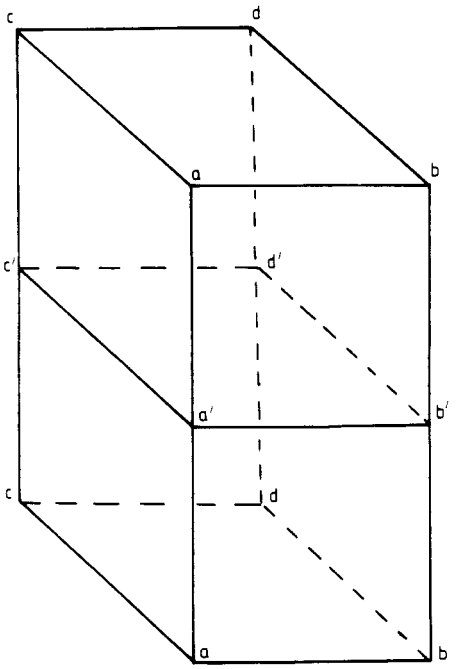

Figure 8. A pillar of two cubes in the two-layer lattice.

The function W_p is given in table 2. We see that it depends on the eight spins $a, b, \ldots, h$ only via the four products $aba'd'$, $abc'd'$, $acb'd'$ and

$$\mu = adb'c'. \tag{3.2}$$

We now set

$$a' = aa'', \qquad b' = bb'', \qquad c' = cc'', \qquad d' = dd'' \tag{3.3}$$

Table 2. Values of the pillar weight functions W_p: μ is defined by (3.2).

$aba'd'$	$abc'd'$	$acb'd'$	W_p
+	+	+	$(P_0-\mu Q_0)^2$
−	+	+	R_1R_0
+	−	+	R_2R_3
+	+	−	R_3R_2
+	−	−	$(P_1+\mu Q_1)^2$
−	+	−	$\mu(P_2+\mu Q_2)^2$
−	−	+	$\mu(P_3+\mu Q_3)^2$
−	−	−	R_0R_1

so that a'', b'', c'', d'' are the products of the spin pairs on the vertical edge (e.g. $a''=aa'$). Then W_p is a function only of $a''d''$, $tc''d''$, $tb''c''$ where

$$t = abcd. \tag{3.4}$$

For the moment, regard as fixed all the vertical spin-pair products (such as a'', b'', c'', d'') throughout the lattice. We still want to sum over the spins in the top layer, such as a, b, c, d. This gives us a two-dimensional 'interactions-round-a-face' model, but a very simple one: the four spins a, b, c, d round a typical face interact only via their product $abcd$.

This makes it easy to sum over the top spins. We look down from above on the top layer and label the rows by $i=1,\ldots,m$ and the columns by $j=1,\ldots,n$. (Thus $\mathscr{L}$ has $N=2nm$ sites.) We suppose the spin a in figure 8 is in position (i,j) and define

$$t_{ij} = abcd, \tag{3.5}$$

so that t_{ij} is the four-spin product for the face. We do this for all faces.

We can eliminate the site-spins a, b, c, d, ... in favour of the face spins t_{ij}. Strictly, we should note that there are 2^{m+n-1} a-spin configurations to each t-spin configuration (negating all a-spins in a row or column leaves the t-spins unchanged). Also, the t-spins should satisfy the restrictions that $t_{i1}t_{i2}\ldots t_{in}$ and $t_{1j}t_{2j}\ldots t_{mj}$ be unity for all i, j. However, these are boundary effects and should not affect the partition function per site when m and n are large, so we shall simply replace the a-spin summation by one over the t-spins.

The spin t_{ij} enters only the weight W_p of the pillar (i,j), so for each pillar we can sum immediately over the t-spin and obtain a combined Boltzmann weight. This will depend on the four spin pairs a'', b'', c'', d'' round the pillar, so can be written as

$$W_c(a'', b'', c'', d'') = \sum_t W_p, \tag{3.6}$$

where W_p is given as a function of a'', b'', c'', d'', t by table 2 and (3.2)-(3.4).

It is also true that W_r is unchanged by negating a'', b'', c'', d'', i.e.

$$W_c(-a'', -b'', -c'', -d'') = W_c(a'', b'', c'', d''). \tag{3.7}$$

If we now regard $a'',\ldots,d''$ as associated with the sites of a square lattice, then it follows that we have a square-lattice eight-vertex model in a field (Baxter 1982,

pp 202-8), with weights

$$\omega_1 = W_c(+,+,+,+), \qquad \omega_2 = W_c(+,-,-,+), \qquad \omega_3 = W_c(+,-,+,-),$$
$$\omega_4 = W_c(+,+,-,-), \qquad \omega_5 = W_c(+,-,+,+), \qquad \omega_6 = W_c(+,+,-,+), \tag{3.8}$$
$$\omega_7 = W_c(+,+,+,-), \qquad \omega_8 = W_c(-,+,+,+).$$

Evaluating these weights from table 2 and (3.2)-(3.8), we obtain

$$\omega_1 = (P_0 - Q_0)^2 + (P_1 - Q_1)^2, \qquad \omega_2 = (P_1 + Q_1)^2 + (P_0 + Q_0)^2,$$
$$\omega_3 = -(P_3 - Q_3)^2 + (P_2 + Q_2)^2, \qquad \omega_4 = -(P_2 - Q_2)^2 + (P_3 + Q_3)^2, \tag{3.9}$$
$$\omega_5 = \omega_6 = 2R_2R_3, \qquad \omega_7 = \omega_8 = 2R_0R_1.$$

For arbitrary values of the P_i, Q_i, R_i (or even for the values given in (2.1), with u arbitrary), these are the weights of an eight-vertex model in a field. This model has not been solved, and is not in general critical. However, for the Zamolodchikov model the parameters P_i, Q_i, R_i are given by (3.11)-(3.13) of Baxter (1983). They satisfy the relations

$$P_iQ_i = P_0Q_0, \qquad i = 1, 2, 3 \tag{3.10}$$

$$P_0^2 + Q_0^2 = \sum_{i=1}^{3} (P_i^2 + Q_i^2) \tag{3.11}$$

$$R_0^2R_1^2 + R_2^2R_3^2 = (P_1^2 + Q_1^2 + P_2^2 + Q_2^2)(P_1^2 + Q_1^2 + P_3^2 + Q_3^2). \tag{3.12}$$

(In particular, these relations are satisfied for the isotropic case, when P_i, Q_i, R_i are given by (2.1) and (2.2).)

Using these relations, it follows from (3.9) that

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8. \tag{3.13}$$

This is precisely the condition for the model to become the free-fermion model, which is exactly solvable (Fan and Wu 1970). Further, the weights also satisfy the restriction

$$\omega_2 = \omega_1 + \omega_3 + \omega_4, \tag{3.14}$$

which is the condition for the model to be critical (equation (34) of Fan and Wu 1970).

Thus the two-layer Zamolodchikov model is equivalent to the critical planar free-fermion model. It may well be that the Zamolodchikov model is critical for any (even) finite number of layers. This would fit with previous two-dimensional exact solutions: in these the star-triangle relations usually lead to a parametrisation of the weights in terms of elliptic functions, and the model becomes critical when these elliptic functions reduce to trigonometric ones. We can regard any finite-height Zamolodchikov model as two-dimensional (with a 'pillar' of spins per site): the tetrahedron relations then guarantee that the star-triangle relations of this two-dimensional model are satisfied. They involve trigonometric rather than elliptic functions, as in the critical two-dimensional models.

We can evaluate the partition function of the isotropic two-layer Zamolodchikov model (with P_i, Q_i, R_i given by (2.1) and (2.2)), using equations (16) and (17) of Fan and Wu (1970). This gives for the partition function per site of $\mathscr{L}$ (the two-layer lattice)

$$\kappa = 2(2^{1/2} - 1)\, e^{2G/\pi} = 1.484\ldots \tag{3.15}$$

where $G = 0.915\,965\ldots$ is Catalan's constant. Curiously this differs from the three-dimensional value (2.44) only by a factor of $2^{1/4}$.

4. Conclusions

The three-dimensional Zamolodchikov model permits a solution of the tetrahedron relations just as various two-dimensional models (e.g. the Ising, eight-vertex, hard hexagon and critical Potts models; see Baxter 1982) permit solutions of the corresponding star-triangle relations. From experience with the two-dimensional cases, we suspect that if such a model only permits a solution at a special temperature, and if the solution involves trigonometric (rather than elliptic) functions, then the model is critical (or perhaps at a first-order transition, as in the two-dimensional Potts model for $q > 4$).

This suggests that the Zamolodchikov model is critical. This conclusion is consistent with the numerical calculations described in § 2, and with the two-layer case examined in § 3.

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